II. Systems Analysis

SYSTEMS DIVISION

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A. Relationship Between the Averaging Method of Krylov-Bogoliubov and an Approximate First Integral of Motion, *H. Lass*

We consider first the nonlinear equation

$$\frac{dx}{dt} = \mu f(x,t),\ \mu f << 1 \tag{1}$$

with $f(x,t)$ periodic in t, say of period 2π, and we assume that $f(x,t)$ has a Fourier series development in the argument, t.

If $\Omega(x,t) =$ constant is a first integral of Eq. (1), then

$$\mu f(x,t)\frac{\partial \Omega}{\partial x} + \frac{\partial \Omega}{\partial t} = 0 \tag{2}$$

Now let (omitting μ^2 and higher order terms)

$$\Omega(x,t) = F(x) + \mu G(x,t) \tag{3}$$

so that Eq. (2) becomes

$$\mu f(x,t)\left[F'(x) + \mu \frac{\partial G}{\partial x}\right] + \mu \frac{\partial G}{\partial t} = 0 \tag{4}$$

Omitting the term containing μ^2 yields

$$f(x,t)\,F'(x) + \frac{\partial G}{\partial t} = 0$$

$$G(x,t) = -F'(x)\int_0^t f(x,\tau)\,d\tau \tag{5}$$

and

$$\Omega(x,t) = F(x) - \mu F'(x)\int_0^t f(x,\tau)\,d\tau = F(x_0) \tag{6}$$

with $F(x)$ arbitrary, $x = x_0$ at $t = 0$. Eq. **(6)** yields $x = \phi(t,\mu)$, and $F(x)$ will be chosen such that $x = \phi(t,\mu)$ will satisfy Eq. *(1)* within the order of μ^2.

Before determining $F(x)$ let us note that if $y = x(t + 2\pi)$ then

$$F(y) - \mu F'(x) \int_0^{t+2\pi} f(x,\tau)\, d\tau = F(x_0) \tag{7}$$

within the order of μ^2 since x is slowly varying.

From Eqs. **(6)** and (7) we have

$$F'(x)\left[\Delta x - \mu \int_0^{2\pi} f(x,\tau)\, d\tau\right] = 0 \tag{8}$$

with $\Delta x = x(t + 2\pi) - x(t)$. Thus

$$\frac{\Delta x}{2\pi} \approx \frac{\mu}{2\pi}\int_0^{2\pi} f(x,\tau)\, d\tau$$

$$\frac{dx}{dt} \approx \frac{\mu}{2\pi}\int_0^{2\pi} f(x,\tau)\, d\tau \tag{9}$$

Eq. (9) is the Krylov-Bogoliubov equation for dx/dt which can be obtained from Eq. **(1)** by replacing $f(x,t)$ by its mean value in time.

Returning to Eq. **(6)** we obtain

$$F'(x)\left(\frac{dx}{dt} - \mu f(x,t)\right) - \mu \frac{dx}{dt}\left[F''(x)\int_0^t f(x,\tau)\, d\tau + F'(x)\int_0^t \frac{\partial f}{\partial x}\, d\tau\right] = 0 \tag{10}$$

by differentiating with respect to t.

The integrals which occur in Eq. (10) may contain secular terms *so* that we will not omit the $\mu\dot{x} = \mu^2 f$ terms in Eq. (10). Now

$$f(x,t) = a_0(x) + \sum_1^\infty \left[a_n(x)\sin(nt) + b_n(x)\cos(nt)\right]$$

$$\frac{\partial f}{\partial x} = a_0'(x) + \sum_1^\infty \left[a_n'(x)\sin(nt) + b_n'(x)\cos(nt)\right] \tag{11}$$

so that if we choose $F(x)$ such that

$$F''(x)\, a_0(x) + F'(x)\, a_0'(x) = 0 \tag{12}$$

it will follow that $x = \phi(t,\mu)$ will satisfy Eq. **(1)** within the order of μ^2 terms.

Eq. (12) yields

$$F(x) = \int \frac{dx}{a_0(x)} \tag{13}$$

with $a_0(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,\tau)\, d\tau$. Once $F(x)$ is determined, Eq. (6) yields $x = \phi(t,\mu)$.

Let us apply this technique to the equation

$$\dot{x} = 2\mu x \sin^2 t \tag{14}$$

We have $a_0(x) = \frac{2x}{2\pi}\int_0^{2\pi} \sin^2\tau\, d\tau = x$, $F(x) = \ln x$, and

$$\ln x - 2\mu \int_0^t \sin^2\tau\, d\tau = \ln x_0$$

$$x = x_0 e^{\mu\left[t - \frac{\sin 2t}{2}\right]} \tag{15}$$

which is the exact solution. The K.B. solution is $x = x_0 e^{\mu t}$

The generalization to a system of equations of the form

$$\frac{dx^i}{dt} = \mu f^i(x,t),\ \mu f^i \ll 1,\ i = 1,2,\cdots n \tag{16}$$

is not very difficult. Let

$$\Omega_i = F_i(x) + \mu G_i(x,t) = \text{constant} = c_i \tag{17}$$

for $i = 1,2,\cdots$ n, be a set of first integrals omitting higher order terms in μ.

Then

$$f^\alpha(x,t)\frac{\partial Fi}{\partial x^\alpha} + \frac{\partial Gi}{\partial t} = 0$$

$$G_i(x,t) = -\frac{\partial Fi}{\partial x^\alpha}\int_0^t f^\alpha(x,\tau)\, d\tau \tag{18}$$

so that

$$F_i(x) - \mu \frac{\partial Fi}{\partial x^\alpha} \int_0^t f^\alpha(x,\tau)\, d\tau = F_i(x_0), i = 1,2,\cdots, n \qquad (19)$$

is a set of first integrals with $F_i(x)$ arbitrary.

We choose the $F_i(x)$ in such a way that Eq. **(19)** will yield the set $xi = x_i(t,\mu)$ satisfying Eq. **(16)** within the order of μ^2 terms.

Differentiating Eq. **(19)** with respect to t yields

$$\frac{\partial Fi}{\partial x^\alpha}(\dot{x}^\alpha - \mu f^\alpha) - \mu \dot{x}^\beta \left[\frac{\partial^2 Fi}{\partial x^\beta \partial x^\alpha} \int_0^t f^\alpha(x,\tau)\, d\tau + \frac{\partial Fi}{\partial x^\alpha} \int_0^t \frac{\partial f^\alpha}{\partial x^\beta}\, d\tau \right] = 0 \qquad (20)$$

Let

$$f^\alpha(x,t) = a^\alpha(x) + \sum_1^\infty \left[a_n^\alpha \sin nt + b_n^\alpha \cos nt \right]$$

If we choose $F_i(x^1,x^2,\cdots,x^n)$ such that

$$a^\alpha(x) \frac{\partial^2 Fi}{\partial x^\beta \partial x^\alpha} + \frac{\partial a^\alpha}{\partial x^\beta} \frac{\partial Fi}{\partial x^\alpha} = 0 \qquad (21)$$

then $\dot{x}^\alpha = \mu f^\alpha$ neglecting μ^2 terms, since Eq. **(21)** eliminates the secular terms.

A first integral of Eq. **(21)** is given by

$$a^\alpha \frac{\partial Fi}{\partial x^\alpha} = k_i = \text{constant}, i = 1,2,\cdots, n \qquad (22)$$

It is known that we can obtain the $F_i(x^1,x^2,\cdots,x^n)$ which satisfy Eq. (22) if we can solve the system

$$\frac{dx^1}{a^1(x)} = \frac{dx^2}{a^2(x)} = \cdots\cdots\cdots = \frac{dx^n}{a^n(x)} = \frac{dF_i}{k_i} \qquad (23)$$

for n first integrals. This is equivalent to solving the system

$$\frac{dx^i}{dt} = a^i(x) = \frac{1}{2\pi} \int_0^{2\pi} f^i(x,\tau)\, d\tau \qquad (24)$$

The solutions obtained from the method outlined above correspond to the Bogoliubov-Mitropolsky first approximate solution.

As a simple example we consider the nonlinear system

$$\dot{x} = \mu y^2$$
$$\dot{y} = \mu x \cos t \qquad (25)$$

with $x = x_0, y = y_0 \neq 0$, at t = 0. Eq. **(22)** becomes

$$y^2 \frac{\partial Fi}{\partial x} = k_i \qquad (26)$$

which yields $F_1 = y (k_1 = 0)$, and $F_2 = x/y^2$ $(k_2 = 1)$.

Eq. **(19)** now yields

$$y - \mu x \sin t = y_0$$
$$\frac{x}{y^2} - \mu t + \frac{2\mu x^2}{y^3} \sin t = \frac{x_0}{y_0^2} \qquad (27)$$

which can be solved for $x = x(t,\mu)$, $y = y(t,\mu)$.

We turn our attention now to the nonlinear equation

$$\frac{d^2x}{dt^2} + x = \mu f\left(x, \frac{dx}{dt}\right), \mu f \ll 1 \qquad (28)$$

or

$$\dot{x} = y$$
$$\dot{y} = -x + \mu f(x,y) \qquad (29)$$

If we let

$$x = A(t) \sin t + B(t) \cos t$$
$$y = A(t) \cos t - B(t) \sin t \qquad (30)$$

it is a simple matter to show that Eq. **(29)** yields

$$\frac{dA}{dt} = \mu \cos t\, f(A \sin t + B \cos t, A \cos t - B \sin t)$$
$$\frac{dB}{dt} = -\mu \sin t\, f(A \sin t + B \cos t, A \cos t - B \sin t) \tag{31}$$

We are now in a position to solve for $A(t)$, $B(t)$ from the above considerations

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B. A Note on Stratonovich's Equation for the Conditional Probability Distribution of the State of a Nonlinear System, with Application to Nonlinear Filtering,

S. R. McReynolds

1. Introduction

A fundamental problem of interest in engineering is as follows. Let $x(t) = (x_1(t), \cdots, x_n(t))$ denote the state vector of a system which satisfies the stochastic differential equation:

$$\dot{x}_i = f_i(x,t) + v_i(t) \qquad (i = 1,n) \tag{1}$$

$f_i(x,t)$ $(i = 1,n)$ are generally nonlinear functions.

$v_i(t)$ $(i = 1,n)$ is white noise with the statistical properties

$E(v_i(t)) = 0; E(v_i(t)\, v_j(T)) = Q_{ij}(t)\delta(t - T) \qquad (i,j = 1,n)$

$\delta(t - T)$ is the Dirac delta function. The system is observed by a set of measurements.

$$z_j(t) = h_j(x,t) + w_j(t) \qquad (j = 1,p) \tag{2}$$

$h(x,t)$ $(j = 1,p)$ are generally nonlinear functions; $w_j(t)$ $(j = 1,p)$ is a white noise vector with statistics given by $E(w_j(t)) = 0$; $E(w_i(t)\, w_j(T)) = R_{ij}(t)\delta(t - T) \qquad (i,j = 1,p)$ For the purposes of this paper, it will be assumed that $v_i(t)$ and $w_i(t)$ are statistically independent. An initial probabilities distribution of x is given explicitly

$$P(x_0,t_0) = \text{given} \tag{3}$$

The problem is to find the function $P(x,t/Z_t)$, expressing conditional probability of the state x at time t, given all past measurements $z(t)$ over the interval (t_0,t) and an initial condition **(3).**

The problem without measurements was originally investigated in the study of Brownian motion. The classical result is the Fokker-Planck equation:

$$0 = \frac{\partial P}{\partial t} + \frac{\partial}{\partial x_i}(Pf_i) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(Q_{ij}P) \tag{4}[1]$$

This is a second-order, parabolic partial differential equation by which the probability distribution $P(x,t/Z_t)$ may be computed forward in time from the given initial condition.

The problem with measurements was solved recently by Stratonovich (Ref. 2), who obtained the partial differential equation:

$$0 = \frac{\partial P}{\partial t} + \frac{\partial}{\partial x_i}(Pf_i) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(Q_{ij}P) + P[\hat{H}(z(t),t) - H(x,z(t),t)] = 0 \tag{5}$$

where

$$H(x,z(t),t) = z(t)(R^{-1})_{ij}h_j(x,t) - \frac{1}{2}h_i(x,t)(R^{-1})_{ij}h_j(x,t) \tag{6}$$

$$\hat{H}(z(t),t) = \int_{-\infty}^{+\infty} H(x,z(t))P(x,t/Z_t)\, dx \tag{7}$$

Note that Eq. (6), which we shall refer to as Stratonovich's equation, has some additional terms. Kushner (Ref. **3)** has contested these terms. Kushner claims the proper equation should be:

$$0 = \frac{\partial P}{\partial t} + \frac{\partial}{\partial x_i}(Pf_i) - \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}(Q_{ij}P) - (z_i - \hat{h}_i)(R^{-1})_{ij}(h_j - \hat{h}_j)P = 0 \tag{8}$$

In this paper we show that Stratonovich's equation, namely, Eq. (5), is the correct equation. Eq. (5) is rederived. It is then shown that in the case when $f(x,t)$ and $h(x,t)$ are linear in x the solution of Stratonovich's equation yields the Kalman filter, whereas Eq. (8) does not.

A recent paper of Kulman (Ref. **4)** also defended Stratonovich's equation. However, Kulman infers, without

[1] Summation over repeated subscripts is implied.